STAT0041: Stochastic Calculus

Lecture 11 - Itô Formula

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Key concepts:

- *Itô formula.*

11.1 Itô process

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathrm{P})$ be a filtrated probability space satisfies usual conditions, B_t be a $\mathscr{F}_t$-adapted Brownian motion.

Definition 11.1 (Itô process) *A stochastic process (X_t) that can be written as*

$$X_t = X_0 + \int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}B_s, \tag{11.1}$$

where $u, v \in \mathscr{L}_T^2$ is called a Itô process.

Formally, we can write (11.1) in differential form

$$\mathrm{d}X_t = u_t \mathrm{d}t + v_t \mathrm{d}B_t. \tag{11.2}$$

An extension of the previous definition is when B is a m-dimensional Brownian motion, X_0 is a n-dimensional random vector v is a $n \times m$ matrix and u is a n-dimensional vector. Then, multi-dimension Itô process (X_t) is a n-dimensional vector with components given by

$$X_t^i = X_0^i + \int_0^t u_s^i ds + \sum_{j=1}^m \int_0^t v_s^{i,j} dB_s^j, \quad i = 1, ..., n. \tag{11.3}$$

11.2 Itô formula

If we only could compute Itô integrals using their definition, the concept will be of limited applicability. We hope Itô integrals can be calculated following some rules like Calculus.

Recall Newton-Leibniz formula, which also known as fundamental theorem of calculus

$$F(b) - F(a) = \int_a^b F'(x) \mathrm{d}x$$

$$g(f(b)) - g(f(a)) = \int_{f(a)}^{f(b)} g'(z)\mathrm{d}z = \int_a^b g'(f(x))f'(x)\mathrm{d}x = \int_a^b g'(f(x))\mathrm{d}f(x).$$

Question is whether a similar formula exists for Itô integral, such that

$$g(B_t) - g(B_0) = \int_0^t g'(B_s)\mathrm{d}B_s$$

The answer is NO. We have calculated that $\int_0^t B_s \mathrm{d}B_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$, for $g(x) = x^2$

$$B_t^2 - B_0^2 = 2\int_0^t B_s \mathrm{d}B_s + t \neq \int_0^t g'(B_s)\mathrm{d}B_s.$$

While we have following Itô formula.

Theorem 11.2 (Itô formula) *Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ are continuous functions and*

$$X_t = X_0 + \int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}B_s$$

be an Itô process. Then the process $Y_t = f(t, X_t)$ is still an Itô process and

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t) \cdot (dX_t)^2,$$

where $(dX_t)^2$ follows the Itô product rule

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \quad dB_t \cdot dB_t = dt.$$

Equivalently,

$$f(t, X_t) - f(0, X_0) = \int_0^t \left\{ \frac{\partial f}{\partial t}(s, X_s) + u_s \frac{\partial f}{\partial x}(s, X_s) + \frac{1}{2}v_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) \right\} ds + \int_0^t v_s \frac{\partial f}{\partial x}(s, X_s) dB_s, \tag{11.4}$$

Example 11.3 *Calculate*

(1) $\int_0^t B_s \mathrm{d}B_s$.

(2) $\int_0^t s \mathrm{d}B_s$

Corollary 11.4 (Integration by parts formula) *Suppose f is continuous and of bounded variation in $[0, t]$. Then*

$$\int_0^t f(s)dB_s = f(t)B_t - \int_0^t B_s df_s.$$

Theorem 11.5 (Multi-dimension Itô formula) *Let B be a m-dimensional Brownian motion and X a n-dimensional Itô process such as in* (11.3). *Let f be a $\mathbb{R}^d$- valued function with twice continuously differentiable components. Then the process $Y_t = f(t, X_t)$ is still an Itô process and*

$$dY_t^k = \frac{\partial f^k}{\partial t}(t, X_t)dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X_t)dX_t^i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)dX_t^i dX_t^j, \quad k = 1, ..., d. \tag{11.5}$$